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A harmonic family of dielectric flow solutions with maximal supersymmetry

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Abstract: We construct a new harmonic family: dielectric flow solutions with maximal supersymmetry in eleven-dimensional supergravity. These solutions are asymptotically $AdS_4 \times S^7$, while in the infra-red the M2 branes are dielectrically polarized into M5 branes. These solutions are holographically dual to vacua of the mass deformed theory on M2 branes. They also provide an interesting insight on the supergravity solutions sourced by giant gravitons, allowing one to see how supergravity solves the giant graviton puzzle.

Keywords: AdS-CFT and dS-CFT Correspondence, Supersymmetry and Duality, Supergravity Models.

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1. Introduction

Supersymmetry in the presence of branes is a well-studied subject, and there are some rather general rules about how supersymmetry is broken by combinations of different kinds of branes. The simplest of these rules involves harmonic distributions of intersecting branes in which the supersymmetry is reduced by each set of branes through a projection condition on the supersymmetry parameter. The solution of [1] was thus rather unexpected in that it represents an M-theory flow solution involving both M2 and M5 branes and yet preserves *sixteen* supercharges. This solution is asymptotic to $AdS_4 \times S^7$, and hence to a standard harmonic M2 brane solution with sixteen supersymmetries, and yet it has non-trivial magnetic fluxes that dielectrically deform the M2 branes into M5 branes in the interior. It does this while preserving all the supersymmetries, indeed it performs a dielectric rotation on the standard supersymmetry condition that defines the supersymmetry for pure M2 branes.

This new solution was obtained by "lifting" a four-dimensional gauged supergravity solution, and while such methods have proven very powerful, they are limited in two significant respects: (i) Such lifts tend to be rather special examples corresponding to higher levels of global symmetry, or very symmetric brane distributions, (ii) The lifting process does not yield much insight into how to generalize the solutions and see if there are entire families of such solutions. Our purpose here is to show that the solution in [1] is part of a large family of new solutions, and that this family is characterized by a new form of "harmonic" distribution. Indeed, it was not realized in [1] that the new solution had an underlying harmonic structure. Moreover, given the recent work on algebraic Killing spinors [2], it is surprising that the new family of solutions is governed by a *linear* partial differential equation.

From the field theory perspective, we will see that the backgrounds we construct here are the duals of vacua of the mass-deformed theory on M2 branes. All these vacua have been studied perturbatively in [3] and found to contain M2 branes polarized into M5 branes. They are the eleven-dimensional analogues of the Polchinski-Strassler flow.

A first hint that there is indeed a whole family of solutions can be obtained by probing the solution of [1]. It turns out that the right probes are not M2 branes, as one might naively expect, but M2 branes polarized into M5 branes [3]. By performing a probe analysis of this background, it is possible to see that there is a configuration space on which the potential for the probes is flat. One should thus be able to deform the particular solution of [1] by simply adding more and more branes along this moduli space. It is interesting therefore to try to construct this family of solutions directly. Although this might appear to be quite a daunting task, several methods have been recently developed [2] that can be used to find this solution: The basic idea is to find the Killing spinors, and then use them to build the metric and forms.

We should remark that, *a priori*, it is quite amazing that such solutions even exist. Indeed, it seems to be commonly assumed that the only M-theory static[1] supergravity solutions with so much supersymmetry are Coulomb branch configurations of the branes of M-theory or their duals. The solution of [1], and the generalizations presented here are the first static solutions with maximal supersymmetry whose building blocks are not usual branes, but are simply dielectric branes [4].

Our presentation, to some degree, reverses the order in which the work was done: We start in section 2 with the Ansatz for the metric, the forms and the Killing spinors. To render the computation manageable we restrict to a family of solutions that preserves an $SO(4) \times SO(4)$ subgroup of the original $SO(8)$ $\mathcal{R}$-symmetry. This means that we allow the solutions to depend upon two "radial" variables, one for each of the two $\mathbb{R}^4$ factors in the $\mathbb{R}^4 \times \mathbb{R}^4 = \mathbb{R}^8$ transverse to the original set of M2 branes. The $SO(4)$'s then act as rotations on each of the $\mathbb{R}^4$'s. This Ansatz enables us to find solutions with arbitrary radial distributions of concentric shells of branes in each of these $\mathbb{R}^4$'s.

The only other known dielectric brane configurations that preserve sixteen supercharges are giant gravitons [8]. In fact, it is not hard to see that by reducing our solutions along the M2 branes one obtains F1 strings polarized into D4 branes, which, when T-dualized along the strings, become momentum waves polarized into D3 branes in an external five-form field strength. Hence, our solutions are dual to the giant graviton solutions in type IIB, in the region near the S^5 equator, where the geometry has an $SO(4) \times SO(4)$

[1] Requiring the solution to be static excludes M-theory giant gravitons and plane waves.

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$\mathcal{R}$-symmetry. It is quite likely that by starting from a more generic Ansatz one could use our methods to obtain the most generic giant graviton solution. However, our solutions describe giant gravitons for a rather large range of parameters, and are therefore very useful in trying to understand their physics. As we will see in section 5, our supergravity analysis indicates that valid giant graviton solutions only exists when the number of D3 branes is smaller than the square root of the number of gravitons, which strongly indicates that the solution to the giant graviton puzzle presented in [11] is correct.

In section 3 we use some of the G-structure equations [5] to simplify the Killing spinor Ansatz, and then in section 4 we complete the solution. In section 5 we explore the physics of the solution when the M2 branes are spread on a shell, and find that near the shell the solution reproduces the M5 branes solution. This links our solutions to giant gravitons, and to the perturbative analysis of [3]. Section 6 contains the brane-probe analysis of the solution of [1], and shows how this solution is related to the linearized solutions of [3]. Section 7 contains some final remarks.

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2. The ansatz

Since we are imposing $\mathrm{SO}(4)\times\mathrm{SO}(4)$ on the $\mathbb{R}^4\times\mathbb{R}^4$ transverse to the branes, we take the metric to have Lorentz frames of the form:

$$\begin{aligned} e^1 &= e^{A_0(u,v)}dt\,; \qquad e^2 = e^{A_0(u,v)}dx\,; \qquad e^3 = e^{A_0(u,v)}dy && (2.1)\\ e^4 &= e^{A_1(u,v)}du\,; \qquad e^5 = e^{A_2(u,v)}dv && (2.2)\\ e^{5+j} &= \frac{1}{2}\,e^{A_3(u,v)}\,u\,\sigma_j \qquad e^{8+j} = \frac{1}{2}\,e^{A_4(u,v)}\,v\,\tau_j\,, && (2.3)\end{aligned}$$

where σ_i and τ_i are the left-invariant 1-forms parameterizing the three-spheres invariant under the action of the $\mathrm{SO}(4)\times\mathrm{SO}(4)$ $\mathcal{R}$-symmetry. Similarly, the most general 3-form potential compatible with the $\mathcal{R}$-symmetry is

$$C = m_0(u,v)dt\wedge dx\wedge dy + m_1(u,v)\,\sigma_1\wedge\sigma_2\wedge\sigma_3 + m_2(u,v)\,\tau_1\wedge\tau_2\wedge\tau_3\,. \qquad (2.4)$$

As in the solution [1], the projector that determines the sixteen supersymmetries is a deformation of the usual M2 brane projector:

$$\begin{aligned}\Pi_0 &= \frac{1}{2}(1+p_1\Gamma^{123}+p_2\Gamma^{45678}+p_3\Gamma^{459\,10\,11}) && (2.5)\\ &= \frac{1}{2}(1+p_1\Gamma^{123}-p_2\Gamma^{1239\,10\,11}+p_3\Gamma^{1236\,7\,8})\,, && (2.6)\end{aligned}$$

where we have used the fact that the product of all the gamma matrices is the identity matrix.

Because only C_{123}, C_{678}, and $C_{9\,10\,11}$ are non-zero, it is not hard to see that the combination

$$M\epsilon = \Gamma^1\delta\psi_1 + \Gamma^6\delta\psi_6 + \Gamma^9\delta\psi_9 \qquad (2.7)$$

is independent of the Maxwell tensors. Indeed, if one drops all the derivative terms from the $\delta\psi_\mu$ in (2.7), the remaining 16×16 matrix turns out to be

$$M_0 = \frac{1}{2}\frac{(1 + u\, {A_0}^{(1,0)} + u\, {A_3}^{(1,0)} + u\, {A_4}^{(1,0)})}{e^{A_1}\, u}\Gamma^4 + \\ + \frac{1}{2}\frac{(1 + v\, {A_0}^{(0,1)} + v\, {A_3}^{(0,1)} + v\, {A_4}^{(0,1)})}{e^{A_2}\, v}\Gamma^5 + \frac{e^{-A_3}}{2u}\Gamma^{678} + \frac{e^{-A_4}}{2v}\Gamma^{9\,10\,11}\,. \tag{2.8}$$

Up to now we have not used the reparametrization invariance available from the definitions of u and v. To fix one of these reparametrizations one can essentially choose either A_3 or A_4, or a combination, at will. To be more precise, one can make a coordinate redefinition $\tilde{u} = \frac{1}{2L} u \,\exp(\Lambda + (A_0 + A_3 + A_4))$, $\tilde{v} = \frac{1}{2L} v\, \exp(-\Lambda + (A_0 + A_3 + A_4))$, for some function $\Lambda(u, v)$, and a constant, L. In the new metric, the coefficient functions will satisfy $\tilde{A}_0 + \tilde{A}_3 + \tilde{A}_4 = \log(4L^2)$, but the change of variables will generate a $d\tilde{u}\, d\tilde{v}$ term. One can then choose the function $\Lambda(u, v)$ so as to eliminate this cross term. One should note that the pure M2 brane solution has $A_3 = A_4 = -\frac{1}{2}A_0$, and in particular $A_0 + A_3 + A_4$ is zero. Thus the foregoing change of variables matches precisely with the desired asymptotic behavior of our solution. We have introduced a scale, L, for later convenience. Dropping the tildes, we therefore fix the coordinate invariance, while preserving the form of the frames in (2.3), by setting:

$$A_0 + A_3 + A_4 = \log(4L^2)\,. \tag{2.9}$$

This simplifies M_0 to:

$$M_0 = \frac{e^{-A_1}}{2\,u}\Gamma^4 + \frac{e^{-A_2}}{2\,v}\Gamma^5 + \frac{e^{-A_3}}{2u}\Gamma^{678} + \frac{e^{-A_4}}{2v}\Gamma^{9\,10\,11}\,. \tag{2.10}$$

Poincaré invariance means that the Killing spinors do not depend upon the coordinates parallel to the brane. Under the $\mathcal{R}$-symmetry the supersymmetries break up into various combinations of SU(2) doublets. This was analyzed in [1], and one finds that the sixteen Killing spinors can be divided into two sets: Half of the spinors are independent of all the sphere coordinates, while the other half depend in a very simple manner upon the sphere coordinates. To separate these two sets one needs to find a second pair of projectors, $\Pi_1^{\pm}$, that project onto these two sets of eight Killing spinors. Given these projectors, the dependence of the spinors upon the sphere directions is completely determined by the Lie derivatives of the spinor, and one finds:

$$d\,\varepsilon + \frac{1}{2}\bigg(\big(\sigma_1\,\Gamma^{78} - \sigma_2\,\Gamma^{68} + \sigma_3\,\Gamma^{67}\big) + \\ + \big(\tau_1\,\Gamma^{10\,11} - \tau_2\,\Gamma^{9\,11} + \tau_3\,\Gamma^{9\,10}\big)\bigg)\,\Pi_1^-\,\varepsilon = 0\,, \tag{2.11}$$

where d is the exterior derivative on the spheres alone, Π_1^- is the projector that annihilates the spinors that are independent of the sphere coordinates, and is the identity matrix on the second set of eight spinors. It is elementary to find this projector Π_1^-: Since the derivatives of the sphere-independent spinors do not enter the gravitino variations in (2.7), M_0 must annihilate these eight spinors. The projector, Π_1^-, must therefore be the

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normalized multiple of M_0. Indeed, one can take $\Pi_1^- = u\, e^{A_1}\, \Gamma_4 M_0$. The condition that this be a projector ($\Pi_1^-\, \Pi_1^- = \Pi_1^-$) means that one must have:

$$\frac{e^{-2A_1}}{4\,u^2} + \frac{e^{-2A_2}}{4\,v^2} = \frac{e^{-2A_3}}{4u^2} + \frac{e^{-2A_4}}{4v^2}\,. \tag{2.12}$$

In order not to further reduce the supersymmetry, the projector Π_1^- should be compatible (i.e. commute) with the projector (2.6). This can only happen if

$$\frac{p_2}{p_3} = \frac{e^{A_4} v}{e^{A_3} u}\,, \tag{2.13}$$

or

$$p_2 = \frac{1}{2}\,\beta\, v\, e^{-A_0+A_4}\,, \qquad p_3 = \frac{1}{2}\,\beta\, u\, e^{-A_0+A_3}\,, \tag{2.14}$$

for, *a priori*, some function, $\beta(u,v)$. We will show below that β is, in fact, a constant. Since p_2 and p_3 are the coefficients of the non-trivial, magnetic components of the gauge field, the parameter, β, represents the strength of the deformation away from the pure M2 brane Coulomb branch. Having fixed p_2 and p_3, the remaining function, p_1, is obtained from requiring that $\Pi_0\,\Pi_0 = \Pi_0$, which implies:

$$p_1^2 + p_2^2 + p_3^2 = 1\,. \tag{2.15}$$

To proceed further it is useful to build in much of what we have obtained so far by introducing new variables B_0, B_1, B_2, B_3 so that (2.9) is automatically satisfied:

$$e^{A_0} = 4L^2 e^{B_0}\,, \tag{2.16}$$
$$e^{A_1} = e^{B_1-\frac{B_0}{2}}\,, \tag{2.17}$$
$$e^{A_2} = e^{B_2-\frac{B_0}{2}}\,, \tag{2.18}$$
$$e^{A_3} = e^{B_3-\frac{B_0}{2}}\,, \tag{2.19}$$
$$e^{A_4} = e^{-B_3-\frac{B_0}{2}}\,. \tag{2.20}$$

3. The Killing spinors and the G-structure

Having determined the projector, it is a straightforward exercise to obtain the eight Killing spinors that are annihilated both by Π and by $\Gamma_4 M_0$. The normalization of these spinors is most easily determined by using properties of the G-structure.

First, if ϵ_i and ϵ_j are any two solutions to $\delta\psi_\mu = 0$, then [5] the following must be a Killing vector of the metric:

$$K^\mu_{ij} \equiv \bar{\epsilon}_i \Gamma^\mu \epsilon_j\,. \tag{3.1}$$

We find that we can generate Killing vectors parallel to the branes, and along the spheres in this manner. The former fix the normalization of the Killing spinors obtained from the projectors, while the latter Killing vectors are proportional to β, and the Killing equation implies that β is constant.

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Another very useful Killing spinor sandwich is

$$\Omega^{ij}_{\mu\nu} \equiv \bar{\epsilon}_i \Gamma_{[\mu}\Gamma_{\nu]}\epsilon_j \tag{3.2}$$

which satisfies [5]:

$$(d\Omega_{ij})_{\mu\nu\lambda} = K^{\sigma}_{ij} F_{\sigma\mu\nu\lambda}\,, \tag{3.3}$$

or in form language

$$d\Omega = i_K F^{(4)}\,. \tag{3.4}$$

If the components, K^ρ, are constant (as the Killing vectors we will use are), and the legs of the form are not along sphere direction, one can integrate the relation above to obtain

$$\Omega = i_K C^{(3)}\,. \tag{3.5}$$

We can now chose a Killing spinor combination that gives a Killing vector whose only non-zero space-time component is $K^3 = -1$. This spinor combination also gives a two-form whose non-zero (space-time) components are:

$$\Omega_{12} = 64\, e^{3\,B_0}\, L^6\, q_0 \tag{3.6}$$

$$\Omega_{34} = -2e^{B_1-B_2} L^2 u\beta \tag{3.7}$$

$$\Omega_{35} = 2e^{B_2-B_1} L^2 v\beta \tag{3.8}$$

where

$$q_0(u,v) \equiv \sqrt{1 - \frac{e^{-3\,B_0+2\,B_3}\, u^2\, \beta^2}{64\, L^4} - \frac{e^{-3\,B_0-2\,B_3}\, v^2\, \beta^2}{64\, L^4}}\,. \tag{3.9}$$

Combining this with (3.5) determines

$$C_{123} = \Omega_{12} = 64\, e^{3\,B_0}\, L^6\, q_0\,. \tag{3.10}$$

In the limit $\beta \to 0$ this reduces to the Coulomb branch solution.

One can now use equation (3.4) and the fact that $F^{(4)}_{45ij} = 0$ to obtain a differential equation for the metric coefficients:

$$\partial_u \Omega_{35} = -\partial_v \Omega_{34} \qquad \Leftrightarrow \qquad \partial_u \left(v\, e^{-(B_1-B_2)}\right) = -\partial_v \left(u\, e^{(B_1-B_2)}\right) \tag{3.11}$$

For the other components of the Maxwell tensor things are a bit more complicated. In particular, since they are along sphere directions, it is a little more subtle to integrate (3.4). We can chose a combination of two Killing spinors that gives rise to a Killing vector with space-time components $K^8 = K^{11} = -\beta$, and we find:

$$\begin{aligned}\Omega = {} & e^{2\,B_3}\, L^2\, u^2\, q_0\, \sigma_1 \wedge \sigma_2 - e^{-2\,B_3}\, L^2\, v^2\, q_0\, \tau_1 \wedge \tau_2 + 2\, L^2 \left(e^{2\,B_3}\, u^2 + e^{-2\,B_3}\, v^2\right)^{-1} \times \\ & \times \Big[u\, (e^{B_1-B_2+2\,B_3}\, u^2 + v^2\, q_0)\, du \wedge \sigma_3 - u^2\, v\, (e^{B_1-B_2+2\,B_3} - q_0)\, dv \wedge \sigma_3 + \\ & + u\, v^2\, (e^{B_1-B_2-2\,B_3} - q_0)\, du \wedge \tau_3 - v\, (e^{B_1-B_2-2\,B_3}\, v^2 - u^2\, q_0)\, dv \wedge \tau_3\Big]\,. \end{aligned} \tag{3.12}$$

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Now we make some heavy use of (3.4), and use the fact that:

$$d\sigma_1 = \sigma_2 \wedge \sigma_3\,, \qquad d\tau_1 = \tau_2 \wedge \tau_3\,, \qquad \text{and cyclic.} \tag{3.13}$$

First we use the fact that the gauge field, F, has no components of the form F_{45ij}. This means that all terms of the form $du \wedge dv \wedge \sigma_3$ and $du \wedge dv \wedge \tau_3$ must vanish in $d\Omega$, and the result is two identities for partial derivatives of functions appearing in (3.12). These identities may be re-cast as integrability conditions for two functions, $g_1(u,v)$ and $g_2(u,v)$, defined by:

$$\begin{aligned}
\partial_u g_1 &= 2\,\left(e^{2\,B_3}\,u^2 + e^{-2\,B_3}\,v^2\right)^{-1}\,u\,\left(e^{B_1-B_2+2\,B_3}\,u^2 + v^2\,q_0\right) && (3.14)\\
\partial_v g_1 &= -2\,\left(e^{2\,B_3}\,u^2 + e^{-2\,B_3}\,v^2\right)^{-1}\,u^2\,v\,\left(e^{B_1-B_2+2\,B_3} - q_0\right) && (3.15)\\
\partial_u g_2 &= 2\,\left(e^{2\,B_3}\,u^2 + e^{-2\,B_3}\,v^2\right)^{-1}\,u\,v^2\,\left(e^{B_1-B_2-2\,B_3} - q_0\right) && (3.16)\\
\partial_v g_2 &= -2\,\left(e^{2\,B_3}\,u^2 + e^{-2\,B_3}\,v^2\right)^{-1}\,v\,\left(e^{B_1-B_2-2\,B_3}\,v^2 + u^2\,q_0\right). && (3.17)
\end{aligned}$$

From (3.14)–(3.17) one can easily see that:

$$\partial_u(g_1+g_2) = 2\,u\,e^{B_1-B_2}\,, \qquad \partial_v(g_1+g_2) = -2\,v\,e^{-(B_1-B_2)}\,. \tag{3.18}$$

The integrability condition for this is precisely (3.11), which is a nice consistency check.

With these definitions, one can then reconstruct F and trivially integrate it to obtain:

$$\begin{aligned}
C^{(3)} &= 32\,e^{3\,B_0}\,L^6\,q_0\,dt\wedge dx\wedge dy - \frac{L^2}{2\,\beta}\,(e^{2\,B_3}\,u^2\,q_0 - g_1)\,\sigma_1\wedge\sigma_2\wedge\sigma_3 + \\
&\quad + \frac{L^2}{2\,\beta}\,\left(e^{-2\,B_3}\,v^2\,q_0 + g_2\right)\,\tau_1\wedge\tau_2\wedge\tau_3\,.
\end{aligned} \tag{3.19}$$

At this point one has obtained as much as one can from the G-structure differential equations for K^μ and Ω, and so one resorts to solving the supersymmetry variations directly. The problem is that the equations are still fairly complicated, and so we make a simplifying assumption motivated by the solution of [1]. The *metric* of that solution was observed to have a discrete symmetry, $\alpha \to -\alpha$, $\theta \to \frac{\pi}{2} - \theta$, and the flux was odd under this symmetry provided that one also interchanged the σ_j and τ_j. In our formulation (see section 2) this symmetry amounts to interchanging the u and v directions. The fact that the flux is odd under this symmetry suggests that the u and v directions must be undergoing essentially identical, but conjugate processes. Indeed, one can make this an exact symmetry of the entire solution if one combines it with an orientation reversal on the M2 branes and both 3-spheres. One of the consequences of this symmetry is that one finds that $B_1(u,v) = B_2(u,v)$, and so we will now impose this condition here. The equations then simplify dramatically.

First, (3.11) is trivially satisfied, and (3.18) reduces to:

$$g_1(u,v) = g(u,v) + u^2 + k_1\,, \qquad g_2(u,v) = -(g(u,v) + v^2 + k_2)\,, \tag{3.20}$$

for some function, $g(u,v)$, and where the k_j are some integration constants. These constants are pure gauge in (3.19), but they will be important in comparing our result to that of [1].

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Equation (2.12) becomes:

$$e^{-2B_1} = (u^2 + v^2)^{-1} \left(u^2 \, e^{2B_3} + v^2 \, e^{-2B_3}\right), \tag{3.21}$$

while equations (3.14)–(3.17) now reduce to:

$$\partial_u g = -2\,(u^2 + v^2)^{-1} \, u \, v^2 \, e^{2B_1} \, (e^{-2\,B_3} - q_0) \tag{3.22}$$
$$\partial_v g = -2\,(u^2 + v^2)^{-1} \, u^2 \, v \, e^{2B_1} \, (e^{2\,B_3} - q_0)\,. \tag{3.23}$$

The strategy is now very simple: One first uses (3.21) to eliminate all appearances of B_1 and its derivatives from the supersymmetry variations. One then solves (3.21), (3.22) and (3.23) for B_0 and B_3 in terms of $\partial_u g$ and $\partial_v g$ (remember that q_0 contains B_0), and then uses this to eliminate all appearances of B_0, B_3, and derivatives, from the supersymmetry variations. As a consequence, every supersymmetry variation can be written in terms of the first and second derivatives of g.

It is then trivial to find the condition on g that solves the entire system, and the rather amazing result is that one has sixteen supersymmetries if and only if g satisfies the *linear* equation:

$$\frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2} - \frac{1}{u}\frac{\partial g}{\partial u} - \frac{1}{v}\frac{\partial g}{\partial u} = 0\,. \tag{3.24}$$

Even more surprisingly, this equation implies that the function $h \equiv u^{-2} v^{-2} g$ satisfies

$$\frac{\partial^2 h}{\partial u^2} + \frac{\partial^2 h}{\partial v^2} + \frac{3}{u}\frac{\partial h}{\partial u} + \frac{3}{v}\frac{\partial h}{\partial u} = \frac{1}{u^3}\,\partial_u(u^3\,\partial_u h) + \frac{1}{v^3}\,\partial_v(v^3\,\partial_v h) = 0\,, \tag{3.25}$$

which is the harmonic equation in $\mathbb{R}^8$!

We have also verified that this solution of the supersymmetry variations solves all the eleven-dimensional equations of motion.

4. The solution

We have found that any function, $h(u, v)$, satisfying (3.25) gives a solution to the eleven-dimensional supergravity equations of motion, preserving sixteen supercharges. However, it is not *a priori* clear that the harmonic function $h(u, v)$ becomes the usual Coulomb branch harmonic function when $\beta \to 0$. We will first find the relation between the non-dielectric harmonic function and the one we have here, and then explore several examples that make the physics of the solutions transparent.

The first example is the harmonic function that has a point source. When $\beta \to 0$, this becomes the usual $AdS_4 \times S^7$ solution. A non-zero β corresponds to a perturbation of the $AdS_4 \times S^7$ solution by a non-normalizable mode. In the dual boundary theory, this perturbation is dual to turning on an operator that gives equal masses to the four chiral multiplets (8 fermions and 8 bosons). In fact this perturbation is none other than the perturbation studied in [3], which causes the M2 branes to polarize into M5 branes.

We can also spread the source of the harmonic function on one of the two 3-spheres in the problem. This gives a solution that looks like an M5 brane when one is near the

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3-sphere. In the UV this solution asymptotes to the pure M2 brane solution. Therefore, this solution is the exact solution sourced by M2 branes polarized into M5 branes, which is dual to one of the vacua of the massive boundary theory. Since the equation satisfied by the master function g is linear, the exact supergravity solutions dual to all the vacua of this theory (containing multiple shells of M2 branes polarized into M5 branes) can be trivially found by superposing the harmonic functions of individual shells.

4.1 The general dielectric solution

The maximally supersymmetric harmonic solutions we found can be expressed in terms of the function g obeying (3.24), using equations (3.9), (3.19), (3.21), (3.22), (3.23):

$$e^{4\,B_3} = \frac{v^2\,\left(2\,L^2\,u^2+v\,g^{(0,1)}-u\,g^{(1,0)}\right)}{u^2\,\left(2\,L^2\,v^2-v\,g^{(0,1)}+u\,g^{(1,0)}\right)} \tag{4.1}$$

$$e^{4\,B_1} = e^{4\,B_2} = \frac{\left(2\,L^2\,u^2+v\,g^{(0,1)}-u\,g^{(1,0)}\right)\,\left(2\,L^2\,v^2-v\,g^{(0,1)}+u\,g^{(1,0)}\right)}{4\,L^4\,u^2\,v^2} \tag{4.2}$$

$$e^{-3\,B_0} = \frac{L^2}{\beta^2}\frac{2\,L^2\,v\,g^{(0,1)}+2\,L^2\,u\,g^{(1,0)}-(g^{(0,1)})^2-(g^{(1,0)})^2}{2\,u\,v\,\sqrt{2\,L^2\,u^2+v\,g^{(0,1)}-u\,g^{(1,0)}}\,\sqrt{2\,L^2\,v^2-v\,g^{(0,1)}+u\,g^{(1,0)}}} \tag{4.3}$$

$$\frac{1}{2}C_{123} = \frac{64\,L^4\beta^2\,u\,v\,\left(-2\,L^2\,u\,v+u\,g^{(0,1)}+v\,g^{(1,0)}\right)}{-2\,L^2\,v\,g^{(0,1)}-2\,L^2\,u\,g^{(1,0)}+(g^{(0,1)})^2+(g^{(1,0)})^2} \tag{4.4}$$

$$\beta C_{678} = g-\frac{L^2\,u\,\left(u^2+v^2\right)\,g^{(1,0)}}{2\,L^2\,v^2-v\,g^{(0,1)}+u\,g^{(1,0)}} \tag{4.5}$$

$$-\beta C_{9\,10\,11} = g-\frac{L^2\,v\,\left(u^2+v^2\right)\,g^{(0,1)}}{2\,L^2\,u^2+v\,g^{(0,1)}-u\,g^{(1,0)}}\,. \tag{4.6}$$

We observe that in order to recover the usual harmonic solution in the limit $\beta\to 0$, the function g should go to zero like β^2. Therefore, when $\beta=0$, the harmonic function which gives the Coulomb branch solution is:

$$Z=\frac{1}{128L^4\beta^2}\,\frac{u\partial_u g+v\partial_v g}{u^2v^2}\,. \tag{4.7}$$

One can easily show that the operator $u\partial_u g+v\partial_v g$ maps harmonic functions to harmonic functions, and so, if g satisfies (3.24), the function Z is harmonic in $\mathbb{R}^8$. Therefore, if one wants to find the solution that is asymptotic to a Coulomb branch solution given by $Z(u,v)$, one needs to invert (4.7) to find g.

For this, it is useful to introduce the new coordinates x,y defined by

$$u=e^{x+y}\,,\qquad v=e^{x-y}\,. \tag{4.8}$$

Equation (4.7) becomes:

$$128L^4\beta^2 Z(x,y)e^{4x}=\frac{\partial g(x,y)}{\partial x} \tag{4.9}$$

and can be integrated straightforwardly to yield the results we quote below (4.13), (5.2).

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In fact, the condition (4.7) only determines $g(u,v)$ up to an arbitrary additive function $g_1(u,v)$ satisfying (3.24) and $u\partial_u g_1 + v\partial_v g_1 = 0$. It is not hard to see that up to an irrelevant additive constant, the only function satisfying these two equations is

$$g_1(u,v) = c\,\frac{u^2 - v^2}{u^2 + v^2}\,. \tag{4.10}$$

Hence, for any value of the constant c the function $g + g_1$ gives a solution to the equations of motion. We should note that all components of the solution depend explicitly on g_1. Thus, given any Coulomb branch solution there exists a one parameter family of dielectric solutions which asymptote to that solution in the limit $\beta \to 0$.

4.2 Coincident branes: the $AdS_4 \times S^7$ flow with no polarized branes

When all the branes are located at a single point, the harmonic function Z is

$$Z = \frac{R^6}{r^6} = \frac{R^6}{(u^2+v^2)^3}\,, \tag{4.11}$$

where R is the AdS radius. Equation (4.7) implies

$$u\partial_u g + v\partial_v g = 128 L^4 \beta^2 R^6 \frac{u^2 v^2}{(u^2+v^2)^3}\,, \tag{4.12}$$

and hence g is given by

$$g = \frac{B u^2 v^2}{(u^2+v^2)^3} + g_1\,, \tag{4.13}$$

where $B \equiv -64 L^4 \beta^2 R^6$

If we set g_1 to zero, the metric coefficients and the forms become:

$$\begin{aligned}
e^{-3\,B_0} &= \frac{L^2\,B\,\left(L^2\left(u^2+v^2\right)^4 + B\,\left(u^4 - u^2\,v^2 + v^4\right)\right)}{\left(u^2+v^2\right)^3\sqrt{3\,B\,v^2\,\left(u^2-v^2\right) + L^2\left(u^2+v^2\right)^4}\,\sqrt{L^2\left(u^2+v^2\right)^4 - 3\,B\,\left(u^4 - u^2\,v^2\right)}\,\beta^2} \\
e^{4\,B_3} &= \frac{3\,B\,v^2\,\left(u^2-v^2\right) + L^2\left(u^2+v^2\right)^4}{L^2\left(u^2+v^2\right)^4 - 3\,B\,\left(u^4 - u^2\,v^2\right)} \\
e^{4\,B_1} &= \frac{\left(3\,B\,v^2\,\left(u^2-v^2\right) + L^2\left(u^2+v^2\right)^4\right)\left(L^2\left(u^2+v^2\right)^4 - 3\,B\,\left(u^4 - u^2\,v^2\right)\right)}{L^4\left(u^2+v^2\right)^8} \\
\frac{1}{2}C_{123} &= \frac{-32\,L^4\left(u^2+v^2\right)^3\left(L^2\left(u^2+v^2\right)^4 - B\,\left(u^4 - 4\,u^2\,v^2 + v^4\right)\right)\,\beta^2}{B\,\left(L^2\left(u^2+v^2\right)^4 + B\,\left(u^4 - u^2\,v^2 + v^4\right)\right)} \\
\frac{1}{2}C_{678} &= \frac{-\left(B\,u^4\,\left(3\,B\,v^2\,\left(u^2-v^2\right) - 2\,L^2\left(u^2+v^2\right)^4\right)\right)}{2\left(u^2+v^2\right)^3\left(L^2\left(u^2+v^2\right)^4 - 3\,B\,\left(u^4 - u^2\,v^2\right)\right)\,\beta} \\
\frac{1}{2}C_{9\,10\,11} &= \frac{B\,v^4\,\left(-2\,L^2\left(u^2+v^2\right)^4 - 3\,B\,\left(u^4 - u^2\,v^2\right)\right)}{2\left(u^2+v^2\right)^3\left(3\,B\,v^2\,\left(u^2-v^2\right) + L^2\left(u^2+v^2\right)^4\right)\,\beta}\,.
\end{aligned} \tag{4.14}$$

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It is not hard to see that this solution is quite sick for very small u and v. Some of the metric coefficients become negative, and the size of the σ and τ circles also becomes zero, which signals very large curvature. We should also note that one cannot remove this singularity by adding g_1 or modifying the coordinates. The function g_1 is much smaller than g in the region where the solution becomes pathological. Moreover, the only harmonic functions with a source at $u = v = 0$ are proportional to $\frac{1}{r^6}$, and so the only way to soften the singularities involve smearing out the branes. We will consider this below.

In the ultraviolet, the leading asymptotic terms of the foregoing expressions can be easily found, remembering that $B \propto \beta^2$:

$$e^{-3\,B_0} \to \frac{L^2\,B}{\left(u^2+v^2\right)^3\beta^2} = 64L^6 Z \tag{4.15}$$

$$e^{4\,B_3} \to 1 \tag{4.16}$$

$$C_{123} \to \frac{-64\,L^4\left(u^2+v^2\right)^3\beta^2}{B} = -Z^{-1} \tag{4.17}$$

$$C_{678} \to -\frac{2B\,u^4}{\left(u^2+v^2\right)^3\beta} \tag{4.18}$$

$$C_{9\,10\,11} \to -\frac{2B\,v^4}{\left(u^2+v^2\right)^3\beta}, \tag{4.19}$$

which is an $AdS_4 \times S^7$ solution perturbed with transverse fluxes, whose magnitude is

$$C_{678} \sim u^4 \beta Z\,. \tag{4.20}$$

These fluxes are exactly the ones found in the analysis of the gravity dual of the mass deformed theory on a large number of coincident M2 branes [3]. They correspond to AdS_4 non-normalizable modes, and are dual to turning on a mass for the four chiral multiplets of this theory. This is therefore the M-theory version of Polchinski-Strassler [6]. As discovered in the perturbative analysis in [3], the transverse flux causes the M2 branes to polarize into M5 branes, exactly as in the IIB case [6]. Moreover, in [6] all the vacua of the gauge theory were matched to brane configurations, and it was argued that the solution with no polarized branes should not be physical, because there is no corresponding gauge theory vacuum. Our analysis confirms that this intuition also extends to the M-theory case. We will come back to the issue of this solution when we discuss giant graviton physics in section 5.

Since the perturbative analysis of the system reveals the possibility of having M2 branes polarized into M5 branes, it is quite natural to expect the solutions to be in the class of solutions obtained here. This is further supported by the probe analysis performed in section 6.

5. M2 branes polarized into M5 branes, and giant graviton physics

In order to find the function $g(u, v)$ that gives the solution corresponding to all the M2 branes polarized into one M5 brane shell, we use the fact that if we take $\beta \to 0$, we expect this solution to become the Coulomb branch solution given by M2 branes smeared on a

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3-sphere. It is therefore natural to expect that at least in a region of space, the function g_0 that gives the Coulomb branch solution to give the full polarized solution. It is however also possible that the desired function, g, also receives corrections at higher order in β: $g = g_0 + \beta^2 g_2 + \beta^4 g_4 + \cdots$.

We start by finding and analyzing the solution to lowest order, and neglecting higher order corrections in β. The harmonic function Z that gives a Coulomb branch distribution of M2 branes on a 3-sphere of radius R_0 is given by

$$Z = \frac{R^6}{\left(R_0^2 - 2\,R_0\,u + u^2 + v^2\right)^{3/2}\left(R_0^2 + 2\,R_0\,u + u^2 + v^2\right)^{3/2}}\,, \tag{5.1}$$

where R is the AdS radius. Using the change of variables (4.8) and inverting (4.7), one finds g:

$$g = D\frac{u^2 - R_0^2 - v^2}{\left(R_0^2 - 2\,R_0\,u + u^2 + v^2\right)^{1/2}\left(R_0^2 + 2\,R_0\,u + u^2 + v^2\right)^{1/2}} + g_1\,, \tag{5.2}$$

where $D = \frac{16L^4R^6\beta^2}{R_0^2}$. It is not hard to check that g satisfies (3.24). We would like to explore whether the solution near the shell has M5 brane form. The contribution of g_1 near the shell is small, so we only explore the solution for $g_1 = 0$. To see the form of the solution near the shell we substitute (5.2) in the solution, and define $x \equiv u - R$. In the near shell region one has $x \ll R$.

We first focus on the combination $2L^2u^2 - ug^{(1,0)} + vg^{(1,0)}$, which appears in the numerator of e^{4B_3}. Near the shell, this is approximately:

$$2L^2u^2 - ug^{(1,0)} + vg^{(1,0)} \approx 2L^2R_0^2 - \frac{DR_0v^2}{(x^2+v^2)^{3/2}}\,. \tag{5.3}$$

For $x, v \sim \frac{D}{R_0L^2}$ this combination becomes negative. This may signify either a coordinate breakdown, or the fact that the naive function g receives corrections at higher order in β. To see the M5 brane behavior it is better to stay away from this region, but remain near the shell. We therefore, work in a region in which

$$\frac{D}{L^2R_0} \ll x\,, \qquad v \ll R_0\,. \tag{5.4}$$

In this region the second term of the right hand side of (5.3) can be ignored.

In addition to (5.3), our solution involves another potentially singular expression in terms of the harmonic function. One can estimate the combination in the numerator of e^{-3B_0} to see that:

$$2\,L^2\,v\,g^{(0,1)} + 2\,L^2\,u\,g^{(1,0)} - (g^{(0,1)})^2 - (g^{(1,0)})^2 \approx \frac{2L^2Dv^2R_0}{(v^2+x^2)^{3/2}} - \frac{D^2v^2}{(v^2+x^2)^2}\,. \tag{5.5}$$

This combination also becomes negative for $x, v \sim \frac{D}{R_0L^2}$, but the singular locus, where it becomes negative is different from that of (5.3). Hence, the singularity is not merely a coordinate artifact, but probably signifies the fact that the harmonic function which gives

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the dielectric brane distribution is not straightforward to obtain starting from the harmonic function which gives the Coulomb branch distribution: There may indeed be higher order corrections in β.

Fortunately, in the regime (5.4) the second term of (5.5) drops out, and all the metric functions are positive. The solution becomes:

$$\begin{aligned} e^{4B_1} &= \frac{DR_0}{2L^2(x^2+v^2)^{3/2}} \\ e^{4B_3} &= \left(\frac{DR_0}{2L^2(x^2+v^2)^{3/2}}\right)^{-1} \\ e^{-3B_0} &= \frac{D^{1/2}L^3}{\sqrt{2}R_0^{3/2}\beta^2(x^2+v^2)^{3/4}}\,. \end{aligned} \tag{5.6}$$

In order for the solution near the shell to match the M5 brane harmonic solution, we need the metric along the branes (in the $1,2,3,\sigma_1,\sigma_2,\sigma_3$ directions) to be proportional to $Z_5^{-1/3}$, and the metric transverse to the M5 branes (in the u,v,τ_1,τ_2,τ_3 directions) to be proportional to $Z_5^{2/3}$, where $Z_5 \sim \frac{Q_5}{(x^2+v^2)^{3/2}}$. This gives four equations for the three functions B_0, B_1, and B_3, which have the solution:

$$e^{4B_1} = Z_5\,, \qquad e^{-4B_3} = Z_5\,, \qquad e^{-6B_0} = Z_5\,. \tag{5.7}$$

Hence, after a trivial rescaling of the coordinates, (5.6) gives the $AdS_7 \times S^4$ metric sourced by a collection of M5 branes at $u = R_0$. The radius of the 4-sphere is

$$R_{S^4}^2 = v^2 e^{-2B_3 - B_0}|_{x=0} = \left(\frac{D}{2\beta}\right)^{2/3}\,, \tag{5.8}$$

giving the number of M5 branes in the shell

$$N_5 = \frac{8L^4R^6\beta}{\pi l_p^3 R_0^2}\,. \tag{5.9}$$

One can also compute this charge by integrating the τ field strenghts on a 4-sphere around the shell. Their leading behavior in the near-shell region is:

$$F_{u\tau_1\tau_2\tau_3} = -\frac{3D}{2\beta}\frac{v^4}{(v^2+x^2)^{5/2}} \tag{5.10}$$

$$F_{v\tau_1\tau_2\tau_3} = \frac{3D}{2\beta}\frac{v^3 x}{(v^2+x^2)^{5/2}}\,. \tag{5.11}$$

We introduce spherical coordinates around the shell $x = \rho\cos\theta,\ v = \rho\sin\theta$ and integrate $F^{(4)}$ on the 4-sphere, to find:

$$(2\pi l_p)^3 N_5 = \int_{S^4} F^{(4)} = \frac{2D}{\beta} \times V_{\Omega_3} = \frac{64\pi^2 L^4 R^6 \beta}{R_0^2}\,, \tag{5.12}$$

in perfect agreement with (5.9).

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Hence both the metric and the forms show the presence of an M5 brane wrapped on a three-sphere at radius R_0. Our solution is therefore the first supergravity solution that exactly describes a situation in which branes are polarized into other branes wrapped on a 3-sphere.

For a given M5 dipole charge N_5 the radius of the polarization shell is

$$R_0^2 = \frac{16\pi l_p N_2 \beta}{N_5}, \tag{5.13}$$

where N_2 is the total number of M2 branes. This formula agrees beautifully with the perturbative analysis in [3], where it was found that in the external fields (4.18), (4.19) the radius of the M5 brane shell is also

$$R_0^2 \sim \beta \frac{N_2}{N_5}. \tag{5.14}$$

Thus, as we have expected, our solutions describe the backgrounds dual to the vacua of the mass deformed M2 brane worldvolume theory. To obtain the dual of the most general such vacuum, corresponding to M2 branes polarized into several concentric M5 shells in both transverse $\mathbb{R}^4$'s, one simply needs to superpose the harmonic functions sourced by the shells.

It is interesting to examine what happens in the region where (5.3) and (5.5) become negative. The fact that these combinations vanish on different loci, most probably indicates the fact that the function g in (5.2) gets corrected near the shell. Even if our solution is not valid for $x, v \sim \frac{D}{R_0 L^2}$, one can argue very strongly that in this region it should have a simple continuation to the precise M5 brane form. Indeed, for a very large region outside $x, v \sim \frac{D}{R_0 L^2}$ the metric and the forms give exactly the $AdS_7 \times S^4$ solution of supergravity. Moreover, we know from the supergravity solutions of dissolved branes [7] that very close to the M5 branes the effect of the dissolved M2 branes becomes very small, and the metric is exactly the M5 brane metric. Hence one should be able to find a set of harmonic corrections, perhaps combined with a simple coordinate re-definition that continues the $AdS_7 \times S^4$ to the region with $x, v \sim \frac{D}{R_0 L^2}$.

The foregoing argument that there should be a smooth transition to the pure M5 brane solution was contingent upon there being a large intermediate region in which the solution did indeed become that of the M5 brane. Such a region only exists if (5.4) is satisfied, and this requires that $R_0^2 \gg D$. In terms of the M5 brane and M2 brane charges, eq. (5.4) is equivalent to:

$$N_2 > N_5^2 \tag{5.15}$$

which is the same as the validity condition of the Polchinksi-Strassler perturbative analysis! When (5.15) is not satisfied, the interior of the solution cannot be interpreted in terms of an M5 brane shell.

5.1 How supergravity solves the giant graviton puzzle

Two of the observations we made in the previous subsection have an important role in the physics of giant gravitons. The first is that a solution for M2 branes polarized into an M5

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brane only exists when the number of M2 brane is greater than the square of the number of M5 branes. The second is that the solution in which the hamonic function is sourced by coincident M2 branes is not physical.

As we have discussed, M2 branes polarized into M5 branes can be related to $AdS_5 \times S^5$ giant gravitons in a region near the S^5 equator. This is seen through a dimensional reduction along the M2 brane direction and a T-duality along the resulting F1 string. As it is well known, giant gravitons in $AdS_5 \times S^5$ correspond to field theory chiral primaries. As discussed in [10, 9] using the $\mathcal{N} = 4$ Super Yang Mills, sphere giants and AdS giants correspond to maximally anti-symmetric and respectively maximally symmetric chiral primaries. This was expanded upon in [11] (by using the auxiliary theory describing the gravitons in their near horizon region), where it was argued that a field theory chiral primary is dual to *two* giant gravitons, but these two dual gravitons have non-overlapping regimes of validity. More precisely, a field theory state corresponding to J/k sphere giant gravitons of angular momentum k each is also dual to k AdS giant gravitons of angular momentum J/k each.

A crucial ingredient in this argument was the fact that these two giant gravitons cannot exist simultaneously. In the equatorial region (which is the region where our solutions are valid), the AdS and sphere giants are related by a $\mathbb{Z}_2$ symmetry interchanging the two SO(4)'s. The fact that the gravitons have non-overlapping ranges of validity translates in the requirement that a giant graviton exists only if the number of D3 branes and the angular momentum satisfy

$$N_3 < \sqrt{J}\,. \tag{5.16}$$

Since the number of D3 branes becomes the M5 brane charge, and J becomes the M2 charge, this condition is precisely the one in equation (5.15).

Another important aspect of solving the giant graviton puzzle is finding the fate of the solution with unpolarized gravitons. This is nothing but the singular solution discussed in section 4. Of course, we have not argued that *all* solutions without giant gravitons are singular. However, the most obvious families of solutions which contain coincident branes are singular. The most likely way to make them non-singular is to smear the harmonic functions; however this precisely corresponds to introducing giant gravitons. Other work on finding supergravity duals of giant graviton systems has appeard in [12].

6. M5 branes with M2 charge as probes for small ζ

In this section we show that the solution obtained in [1] is a part of our family of solutions. We also probe this solution using M2 branes polarized into M5 branes, and show that the probes have a moduli space. This was, in fact, our first indication that the solution in [1] is a member of a family of solutions, and this was the original motivation for this work.

6.1 Obtaining the gauged supergravity solution

It is elementary to recover the solution of [1]. The deformation parameter, β, is related to

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the corresponding parameter, ζ, in [1] via

$$\beta \;\equiv\; \frac{1}{L}\,\sin\zeta\,. \tag{6.1}$$

The relevant change of variables is

$$u = \frac{e^{-\frac{1}{2}\alpha}}{\sqrt{\sinh\alpha}}\,\cos\theta\,, \qquad v = \frac{e^{\frac{1}{2}\alpha}}{\sqrt{\sinh\alpha}}\,\sin\theta\,. \tag{6.2}$$

Note that this is precisely the change of coordinates used in [1] to convert the harmonic flow with $\beta = 0$ to the manifestly harmonic form in terms of u and v.

If one substitutes the metric functions of [1] into (3.22) and (3.23), and converts them into differential equations with respect to α and θ, then they can be trivially integrated to obtain:

$$g = \sin^2\left(\frac{1}{2}\,\zeta\right)\,\cos 2\theta\,, \tag{6.3}$$

It is thus this very simple "master function" that determines the entire solution in [1]. To recover the solution exactly as presented in this reference one must take $k_1 = \sin^2(\frac{1}{2}\zeta)$ and $k_2 = -\sin^2(\frac{1}{2}\zeta)$ in (3.20) so as to make $g_1 \sim \cos^2\theta$ and $g_2 \sim \sin^2\theta$. As we underlined before, the choice of these constants is pure gauge.

Thus, we see that the solution of [1] is a single example of the very large class of solutions obtained here.

6.2 The action of the M2-M5 probes

Because of the complexity of the general solution, we will only perform the analysis when ζ is small. This will be sufficient to exhibit the polarization effects and the moduli space. In this limit the background [1] consists of a distribution of parallel M2 branes, with a very small M5 dipole moment. It is thus natural to use a spherical test M5 brane with a large M2 charge to probe this background.

In order to give an M5 brane an M2 charge one needs to turn on its world-volume self-dual three-form field strength. The M5 action is rather complicated [13]–[15], but it is not intractable. The main complication arises from the self-duality of the 3-form, which implies that this form will always have one leg along the time direction. Hence the lagrangian and the hamiltonian will be different. Moreover, self duality makes finding the hamiltonian rather messy.

Instead of pursuing this route, we use the fact that compactified eleven-dimensional supergravity is type IIA supergravity in ten dimensions. The probe potential is independent of the compactification radius. Thus, it is given by the potential of a spherical D4 brane with large F1 charge, in the background [1] reduced along e^2.

One can give F1 string charge to a D4 brane by turning on its world-volume field strength F_{01}. Suppose that the D4 brane is wrapped on a 3-sphere whose vielbeins are e_{ψ_1}, e_{ψ_2} and e_{ψ_3}. The Born-Infeld lagrangian is simply:

$$L = -e_{\psi_1} e_{\psi_2} e_{\psi_3} e^{-\Phi}\sqrt{(e_0 e_1)^2 - (B_{01} + F_{01})^2}\,. \tag{6.4}$$

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The quantized F1 string charge in a non-trivial background is:

$$\Pi = \frac{\delta L}{\delta F_{01}} = e_{\psi_1} e_{\psi_2} e_{\psi_3} e^{-\Phi} \frac{B_{01} + F_{01}}{\sqrt{(e_0 e_1)^2 - (B_{01} + F_{01})^2}} = N\,. \tag{6.5}$$

This gives

$$\frac{1}{\sqrt{(e_0 e_1)^2 - (B_{01} + F_{01})^2}} = \frac{1}{e_0 e_1} \sqrt{1 + \frac{N^2 e^{2\Phi}}{e_{\psi_1}^2 e_{\psi_2}^2 e_{\psi_3}^2}}\,. \tag{6.6}$$

The hamiltonian is

$$H = \Pi F_{01} - L = e_{\psi_1} e_{\psi_2} e_{\psi_3} e^{-\Phi} \frac{e_0^2 e_1^2 - B_{01}(B_{01} + F_{01})}{\sqrt{(e_0 e_1)^2 - (B_{01} + F_{01})^2}} \tag{6.7}$$

$$= e_0 e_1 \sqrt{e_{\psi_1}^2 e_{\psi_2}^2 e_{\psi_3}^2 e^{-2\Phi} + N^2} - N B_{01}\,, \tag{6.8}$$

and we may think of this as the effective potential of the spherical D4 brane with an F1 charge equal to N.

A few consistency checks: In the background created by a large number of F1 strings, the leading terms in the large N expansion cancel — this is simply a reflection of the fact that parallel fundamental strings do not interact. For $N = 0$ we recover the D4 brane action. For $B = 0$ the masses of the D4 brane and F1 string add in quadratures. Moreover one can obtain this expression for the hamiltonian of a D-brane with fundamental string charge by using S-duality. The action of a D3 brane with D1 brane charge N in a RR background field $C^{(2)}$ is $e_0 e_1 \sqrt{e_2^2 e_3^2 e^{-2\Phi} + N^2 e^{-2\Phi}} - N C_{01}$ and after S-duality becomes exactly the IIB version of (6.8).

Remembering that the eleven-dimensional vielbeins are related to the ten-dimensional ones by:

$$e|_{10} = e^{\Phi/3} e|_{11}\,, \qquad e_2|_{11} = e^{2\Phi/3} \tag{6.9}$$

and using (6.8) it is not hard to reexpress the effective potential as a function of the eleven-dimensional parameters:

$$H_{11} = e_1 e_2 e_3 \sqrt{e_{\psi_1}^2 e_{\psi_2}^2 e_{\psi_3}^2 + N^2} - N C_{012}\,, \tag{6.10}$$

where now the vielbeins are eleven-dimensional.

The hamiltonian of the D4 brane also contains a Wess-Zumino coupling to the background RR-fields and NS-NS fields:

$$H_{WZ}^{\mathrm{D4}} = -C_5 + B_2 \wedge C_3\,. \tag{6.11}$$

To lift this to eleven dimensions it is convenient to act on it with an exterior derivative, and use the IIA equation of motion, $dC_5 = - * F_4 + H_3 \wedge C_3$, to obtain:

$$dH_{WZ}^{\mathrm{D4}} = *F_4 + B_2 \wedge F_4\,. \tag{6.12}$$

When reducing the background [1] to ten dimensions, the B field descends from C_{012}, while F_4 and $*F_4$ descend from the four-form F^t transverse to the M2 branes. Hence, the WZ coupling of a probe M5 brane in the background [1] is:

$$dH_{WZ}^{M5} = *F_4^t + C_3^{(123)} \wedge F_4^t\,. \tag{6.13}$$

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6.3 The probe action

In the limit of large string charge and small ζ, the square roots in (6.10) can be expanded, giving three contributions to the effective potential:

$$
\begin{aligned}
H_1 &= \frac{1}{2N} e_1 e_2 e_3 e_{\psi_1}^2 e_{\psi_2}^2 e_{\psi_3}^2 && (6.14)\\
H_2 &= H_{WZ}^{M5} && (6.15)\\
H_3 &= N(e_1 e_2 e_3 - C_{123}) = N e_1 e_2 e_3 - \frac{N k^3 X_3}{2\sinh^3(\alpha)}\,. && (6.16)
\end{aligned}
$$

Both H_1 and H_3 can be found rather straightforwardly. Suppose that the directions of the 3-sphere are be given by $e^{\psi_i} = \cos\gamma_i e^{\sigma_i} + \sin\gamma_i e^{\tau_i}$. The angles, γ_i thus represent the orientation of the probe along the two geometric 3-spheres. For the moment we will take all the γ_i to be equal: $\gamma_i = \gamma$.

For small ζ,

$$
\begin{aligned}
H_1 &= \frac{k^3 L^6}{4N} \frac{(\cos\theta\cos\gamma e^{-\alpha/2} + \sin\theta\sin\gamma e^{\alpha/2})^6}{\sinh^3\alpha} && (6.17)\\
H_3 &= \frac{N k^3 \zeta^2}{4} \frac{\cos^2\theta e^{-\alpha} + \sin^2\theta e^{\alpha}}{\sinh\alpha}\,. && (6.18)
\end{aligned}
$$

Nevertheless, to find H_2 we need to do a bit of work. We Hodge dualize the forms in [1], compute dH_{WZ} using (6.13), and integrate it to obtain:

$$
\begin{aligned}
H_2 = {} & \frac{-k^3 L^3 \zeta}{2} \frac{\cos^4\theta e^{-2\alpha}}{\sinh^2\alpha}\, dx^1\wedge dx^2\wedge dx^3 \wedge\sigma_1\wedge\sigma_2\wedge\sigma_3 + \\
& + \frac{-k^3 L^3 \zeta}{2} \frac{\sin^4\theta e^{2\alpha}}{\sinh^2\alpha}\, dx^1\wedge dx^2\wedge dx^3\wedge\tau_1\wedge\tau_2\wedge\tau_3\,. \qquad (6.19)
\end{aligned}
$$

To be supersymmetric, the effective potential must come from a superpotential, and hence must have the form of a sum of perfect squares of analytic functions in the fundamental physical parameters. One can see that $H = H_1 + H_2 + H_3$ has such a form only if $\theta = 0$ or $\theta = \pi/2$. For these two angles the effective potential can be conveniently written:

$$
\begin{aligned}
H_{\text{total}} = {} & \frac{k^3 L^6}{4N} \frac{\cos^2\theta e^{-\alpha}}{\sinh\alpha} \left(\cos^2\theta\cos^3\gamma e^{-\alpha} - \zeta\frac{N}{L^3}\right)^2 + \\
& + \frac{k^3 L^6}{4N} \frac{\sin^2\theta e^{\alpha}}{\sinh\alpha} \left(\sin^2\theta\sin^3\gamma e^{\alpha} - \zeta\frac{N}{L^3}\right)^2\,. \qquad (6.20)
\end{aligned}
$$

For general γ_i's, $\cos^3\gamma$ is replaced by $\cos\gamma_1\cos\gamma_2\cos\gamma_3$ and similarly for $\sin^3\gamma$.

This suggests that the only supersymmetric ways to add a brane probe is along the u or v axis ($\theta = 0$ or $\theta = \pi/2$). Moreover, it shows that there is a vanishing effective potential, and thus a non-trivial moduli space, when the orientation angles, γ_i, of the probe appropriately adjust as the radial coordinate, α varies. This moduli space is therefore six-dimensional: The three γ_j with one radius and a constraint yield half of the moduli space, and the other half is given by the rotations of the brane around the 3-sphere that it wraps.

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We end this section by observing that for $\zeta \neq 0$ there is no moduli space for a probe M2 brane unless it carries M5 charge. If the probe has no M5 brane charge then H_1 and H_2 vanish, and the effective potential is simply given by H_3 (6.18). This is manifestly positive, and hence there is no moduli space. This shows that the building blocks of the solution [1] are not usual M2 branes, but M2 branes polarized into M5 branes. Hence, the probe computation illustrates one of the conclusions of section 5, by revealing the dielectric structure of these solutions.

7. Conclusions and future directions

We have used some of the rather powerful Killing spinor methods that have been developed over the past few years [2, 5] to construct a harmonic family of maximally supersymmetric eleven-dimensional supergravity solutions describing M2 branes dielectrically polarized into M5 branes.

All other dielectric supergravity solutions that have been constructed until now [16] contain branes polarized into branes one dimension higher, wrapped on a circle. Polarizations of codimension two or three, which are physically much more interesting, have proven difficult to obtain via usual solution generating techniques. Our solution is the first of this kind.

The physics of our solution is very relevant for illuminating a few rather puzzling phenomena which have appeared in systems containing polarized branes. One of these systems is the giant graviton. Our solutions are dual to giant gravitons near the S^5 equator, and show that only giant gravitons with angular momentum greater than the square of the number of D3 branes are physical. Moreover, our solutions support the fact that Kaluza Klein gravitons with large angular momentum are unphysical. This provides further evidence for the fact that in any parameter regime there exists only one supergravity solution dual to a field theory chiral primary.

Our solutions are the M-theory analogue of the yet to be found type IIB solutions dual to the $N = 1^*$ theory in four dimensions [6]. Many vacua of this theory undergo a confinement–deconfinement phase transition. In supergravity this corresponds to a transition from a configuration of D3 branes polarized into a shell of D5 branes, to D3 branes polarized into a shell of NS5 branes. However, the transition regime is inaccessible to Born-Infeld analysis used in [6]. To describe this regime, one really needs the full supergravity solution. In the M-theory solution constructed here this deconfinement phase transition corresponds to going from an M5 brane shell of charge k on the σ sphere to an M5 brane of charge N/k on the τ sphere as k becomes larger than $\sqrt{N}$. Our class of solutions should capture all intermediate solutions, and thus can be used to understand how this transition takes place. It would be very interesting to find the function g giving the phase transition supergravity solutions, and to use these solutions to learn more about this transition in gauge theory.

More generally, we believe the methods used in this study may well prove useful in finding the exact type IIB duals of the $N = 1^*$ theory. Since these solutions also describe

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concentric shells of polarized branes [6], it is quite likely that they will also be generated by a master function satisfying a linear equation.

Last, but not least, our solutions put to rest the well-established belief that the only maximally supersymmetric static solutions are Coulomb branch distributions of branes. The fact that supergravity allows maximally supersymmetric solutions built out of polarized branes attests to the richness of its physics, and to its fascinating interaction with string theory.

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